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Twisting cocycle for null-plane quantized Poincaré algebra

A I Mudrov†

Department of Theoretical Physics, Institute of Physics, St Petersburg State University, Ulyanovskaya 1, Stary Petergof, St Petersburg, 198904, Russia

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Abstract. The twisting cocycle for a quantized Poincaré algebra $\mathcal{P}(3+1)$ introduced by Ballesteros *et al* is evaluated. The solution is obtained as a specific case of a formulated multidimensional generalization of the non-standard (Jordanian) quantization of $sl(2)$.

1. Introduction

Recently Ballesteros *et al* built a quantum deformation of the Poincaré algebra [1]. The quantization found was generated by a triangular classical r-matrix and, according to Drinfeld's theory [2], should be a twisting of $U(\mathcal{P}(3+1))$. Its universal $\mathcal{R}$-matrix was obtained on the basis of the $\mathcal{T}$-matrix approach [3], and the bicrossproduct structure revealed in [4]. In the present paper we evaluate the twisting 2-cocycle governing the deformation process and thus recover the universal matrix from the well known formula $\mathcal{R} = \tau(\mathcal{F}^{-1})\mathcal{F}$ [2]. Knowledge of the twisting element $\mathcal{F}$ is very important, because it deforms not only the symmetry algebra but also the geometry of the space–time. Twisting of a universal enveloping algebra induces coherent transformations in the modules and allows one to obtain important objects automatically; for example, to construct invariant equations and their solutions [6]. In order to solve the problem we resort to the theory of quantizing Lie algebras with quasi-Abelian dual groups (the semidirect product of two commutative subgroups) [7–9]. In the present paper we first consider a class of algebras along the lines of that theory. That class may be regarded as a direct generalization of the triangular or non-standard deformation of the Borel subalgebra in $sl(2)$ [10, 11]. We find the general expression for twisting elements and universal $\mathcal{R}$-matrices and then apply the technique developed to the specific case of $\mathcal{P}(3+1)$.

2. General consideration

Let $\mathbf{L} = \mathbf{H} \triangleleft \mathbf{V}$ be a Lie algebra splitting into a semidirect sum of its two Abelian subalgebras, with the basic elements $H_i \in \mathbf{H}$ and $X_\mu \in \mathbf{V}$:

$$[H_j, X_\mu] = B^i_{j\mu} H_i.$$

Suppose that its dual algebra $\mathbf{L}^*$ also has the structure of a semidirect sum $\mathbf{H}^* \triangleright \mathbf{V}^*$, which is defined via a commutative set of matrices $(\alpha^i)^\mu_\nu$. To match the consistency condition upon $\mathbf{L}$ and $\mathbf{L}^*$ we require

$$(\alpha^i)^\mu_\nu B^j_{k\mu} = (\alpha^j)^\mu_\nu B^i_{k\mu}.$$

† E-mail address: aimudrov@dg2062.spb.du

There exists a quantization $U_\alpha(\mathbf{L})$ of the universal enveloping algebra $U(\mathbf{L})$ with the relations on the generators [8]

$$[H_j, X_\mu] = \left(\frac{\mathrm{e}^{2\alpha\cdot H} - I}{2\alpha\cdot H}\right)_\mu^\nu B_{j\nu}^i H_i \tag{1}$$

and the coproduct

$$\Delta_\alpha(H_i) = H_i\otimes 1 + 1\otimes H_i \qquad \Delta_\alpha(X_\mu) = (\mathrm{e}^{2\alpha\cdot H})_\mu^\nu \otimes X_\nu + X_\mu\otimes 1. \tag{2}$$

The symbol I stands for the identity matrix $I_\nu^\mu = \delta_\nu^\mu$, and $\alpha\cdot H$ means $\sum_i \alpha^i H_i$. The apparent counit is $\varepsilon(X_\mu) = \varepsilon(H_i) = 0$, and the antipode may readily be found from the coproduct with the use of the defining axioms. Its explicit form is irrelevant to our study.

We are interested only in such $U_\alpha(\mathbf{L})$ which are obtained by the twisting of the classical universal enveloping algebras $U(\mathbf{L})$. The goal of our investigation is to find the explicit form of the element $\mathcal{F}\in U(\mathbf{L})\otimes U(\mathbf{L})$ governing that process and the universal $\mathcal{R}$-matrix of the algebra $U_\alpha(\mathbf{L})$. Unexpectedly, it appears easier to start from $U_\alpha(\mathbf{L})$ rather than from the classical algebra, find a solution Φ to the twist equation, and then return to $U(\mathbf{L})$ (we are going to use the group properties of twisting and $\mathcal{F} = \Phi^{-1}$ in particular [2]).

We seek a solution to the twist equation

$$(\Delta_\alpha\otimes id)(\Phi)\Phi_{12} = (id\otimes\Delta_\alpha)(\Phi)\Phi_{23} \tag{3}$$

in the form

$$\Phi = \exp(r^{i\mu} H_i\otimes X_\mu) \in U_\alpha(\mathbf{L})\otimes U_\alpha(\mathbf{L}).$$

The classical skew-symmetric r-matrix then will be $r = r^{i\mu}(X_\mu\otimes H_i - H_i\otimes X_\mu)$, and the matrices α^i will be expressed through the structure constants of $\mathbf{L}$ by the formula

$$(\alpha^i)_\nu^\mu = \tfrac{1}{2} r^{j\mu} B_{j\nu}^i. \tag{4}$$

Without any loss of generality we suppose r to be non-degenerate, since otherwise we may restrict ourselves to the image $r(\mathbf{L}^*) = r^t(\mathbf{L}^*)\subset\mathbf{L}$, which is a subalgebra in $\mathbf{L}$, and twisting a subalgebra induces that of $\mathbf{L}$ as a whole.

Calculating both sides of equation (3)

$$\begin{aligned}(\Delta_\alpha\otimes id)(\mathrm{e}^{r^{i\mu}H_i\otimes X_\mu})\mathrm{e}^{r^{i\mu}H_i\otimes X_\mu\otimes 1} &= \mathrm{e}^{r^{i\mu}(H_i\otimes 1\otimes X_\mu + 1\otimes H_i\otimes X_\mu)}\mathrm{e}^{r^{i\mu}H_i\otimes X_\mu\otimes 1}\\ &= \mathrm{e}^{r^{i\mu}H_i\otimes 1\otimes X_\mu}\mathrm{e}^{r^{i\mu}H_i\otimes X_\mu\otimes 1 + r^{i\mu}r^{j\nu}H_i\otimes[H_j,X_\mu]\otimes X_\nu}\mathrm{e}^{r^{i\mu}1\otimes H_i\otimes X_\mu}\end{aligned}$$

$$\begin{aligned}(id\otimes\Delta_\alpha)(\mathrm{e}^{r^{i\mu}H_i\otimes X_\mu})\mathrm{e}^{r^{i\mu}1\otimes H_i\otimes X_\mu} &= \exp\left(r^{i\mu}(H_i\otimes(\mathrm{e}^{2\alpha\cdot H})_\mu^\nu\otimes X_\nu + H_i\otimes X_\mu\otimes 1)\right)\\ &\quad\times \mathrm{e}^{r^{i\mu}1\otimes H_i\otimes X_\mu}\end{aligned}$$

and then comparing them with each other and taking into account the commutation properties of the generators (H_i) and (X_μ), we arrive at the condition

$$r^{i\mu}H_i\otimes 1\otimes X_\mu + r^{i\mu}r^{j\nu}H_i\otimes[H_j,X_\mu]\otimes X_\nu = r^{i\mu}H_i\otimes(\mathrm{e}^{2\alpha\cdot H})_\mu^\nu\otimes X_\nu$$

which, in its turn, yields $r^{i\nu}(\delta_\nu^\mu + r^{j\mu}B_{j\nu}(H)) = r^{i\nu}(\mathrm{e}^{2\alpha\cdot H})_\nu^\mu$. The latter is fulfilled provided that $r^{j\mu}B_{j\nu}(H) = (\mathrm{e}^{2\alpha\cdot H})_\nu^\mu - \delta_\nu^\mu$ which holds in view of (1) and (4).

Our next goal is to show that the formula $\Delta(h) = \Phi^{-1}\Delta_\alpha(h)\Phi$, $h\in U_\alpha(\mathbf{L})$, defines the classical comultiplication on the universal enveloping algebra $U(\mathbf{L})$. Due to the non-degeneracy of the r-matrix, we can raise and lower indices: $H^\mu = r^{i\mu}H_i$, $(\alpha_\mu)_\nu^\rho = \alpha_{\mu\nu}^\rho = r_{i\mu}(\alpha^i)_\nu^\rho$. The matrix $(r_{i\mu})$ is the inverse of $(r^{i\mu})$: $r^{i\mu}r_{j\mu} = \delta_j^i$. The relations in $U_\alpha(\mathbf{L})$ take the form

$$[H^\mu, X_\nu] = (\mathrm{e}^{2\alpha\cdot H} - I)_\nu^\mu.$$

With the set of numbers ξ^ν fixed, let us introduce the entities K^μ, defining them as

$$K^\mu = \xi^\nu (I - \mathrm{e}^{-2\alpha\cdot H})^\mu_\nu \tag{5}$$

and evaluate the commutation relations between K^μ and X_ν:

$$[K^\mu, X_\nu] = \xi^\beta (\mathrm{e}^{-2\alpha\cdot H})^\rho_\beta (2\alpha^\mu_{\rho\sigma})(\mathrm{e}^{2\alpha\cdot H} - I)^\sigma_\nu.$$

From the commutativity of the matrices α_μ and the condition $\alpha^\rho_{\mu\nu} = \alpha^\rho_{\nu\mu}$ following from the classical Yang–Baxter equation, we find $\alpha^\mu_{\rho\sigma}(\mathrm{e}^{2\alpha\cdot H} - I)^\sigma_\nu = \alpha^\mu_{\nu\sigma}(\mathrm{e}^{2\alpha\cdot H} - I)^\sigma_\rho$. This is verified by simple induction over powers of the matrix $(\alpha \cdot H)$. Finally, we have

$$[K^\mu, X_\nu] = 2\alpha^\mu_{\sigma\nu} K^\sigma$$

i.e. the classical commutation relations. The coproduct on the new generators is

$$\Delta_\alpha(K^\mu) = K^\mu \otimes 1 + (\mathrm{e}^{-2\alpha\cdot H})^\mu_\nu \otimes K^\nu.$$

With the use of this formula we calculate the twisted coproduct, which comes out as

$$\begin{aligned}
\Delta(K) &= \mathrm{e}^{-H\otimes X}\Delta_\alpha(K)\mathrm{e}^{H\otimes X} = K\otimes 1 + (\mathrm{e}^{2\alpha\cdot H}\mathrm{e}^{-2\alpha\cdot H})\otimes K = K\otimes 1 + 1\otimes K \\
\Delta(X) &= \mathrm{e}^{-H\otimes X}\Delta_\alpha(X)\mathrm{e}^{H\otimes X} = \mathrm{e}^{2\alpha\cdot H}\otimes X + \mathrm{e}^{-H\otimes X}(X\otimes 1)\mathrm{e}^{H\otimes X} \\
&= \mathrm{e}^{2\alpha\cdot H}\otimes X + X\otimes 1 - (\mathrm{e}^{2\alpha\cdot H} - I)\otimes X = 1\otimes X + X\otimes 1.
\end{aligned} \tag{6}$$

We might consider that our goal had been achieved were we sure that the number of independent generators K^μ was the same as the dimensionality of space $\mathbf{H}$. That is not the case in general, however, and it is determined by a particular choice of ξ^μ. In the classical limit we have

$$K^\mu = \xi^\nu (2\alpha\cdot H)^\mu_\nu = [H^\mu, \xi^\nu X_\nu].$$

Thus, while ξ^μ takes all possible values the lineal $\mathrm{Span}(K^\mu)$ fills up the subspace $\mathbf{H}' = [\mathbf{H}, \mathbf{V}] \subset \mathbf{H}$. If that subspace coincides with the whole of $\mathbf{H}$ we may state that, indeed, twisting $U_\alpha(\mathbf{L})$ with the element Φ results in $U(\mathbf{L})$. Let us show that $\dim(\mathbf{H}') < \dim(\mathbf{H})$ if and only if the subalgebra $\mathbf{V}$ and the centre of $\mathbf{L}$ have a non-trivial intersection. Indeed, because of the non-degeneracy of the classical r-matrix there exists an isomorphism between the linear spaces $\mathbf{V}^*$ and $\mathbf{H}$ (the basic elements H^μ and X_μ turn out to be mutually dual). The subspace $\mathbf{H}'$ is less than $\mathbf{H}$ if and only if there is an element $X_0 = \xi^\mu_0 X_\mu \in \mathbf{H}^*$, orthogonal to the entire $\mathbf{H}'$:

$$0 = (X_0, [H^\mu, X_\nu]) = 2\xi^\sigma_0 \alpha^\mu_{\sigma\nu}.$$

Due to the symmetry $\alpha^\mu_{\sigma\nu} = \alpha^\mu_{\nu\sigma}$, the latter expression is simply the matrix of the adjoint representation $\mathrm{ad}(X_0)$ restricted to the subspace $\mathbf{H}$. Let us summarize the results of our study.

Theorem 1. The element $\Phi = \exp(r^{i\mu} H_i \otimes X_\mu) \in U_\alpha(\mathbf{L}) \otimes U_\alpha(\mathbf{L})$ is a solution to the twist equation. Twisting $U_\alpha(\mathbf{L})$ with the help of Φ gives the classical universal enveloping algebra $U(\mathbf{L})$ unless $\mathbf{V}$ contains central elements. The universal $\mathcal{R}$-matrix for $U_\alpha(\mathbf{L})$ is

$$\mathcal{R} = \exp(r^{i\mu} X_\mu \otimes H_i)\exp(-r^{i\mu} H_i \otimes X_\mu). \tag{7}$$

Expression (7) is an apparent generalization of the $\mathcal{R}$-matrix for $U_h(sl(2))$ in the form of Ogievetsky.

It becomes clear from the above study that the algebras $U_\alpha(\mathbf{L})$ covered by the theorem are completely specified by the set of commutative matrices α_μ, satisfying the requirement $(\alpha_\mu)^\sigma_\nu = (\alpha_\nu)^\sigma_\mu$. Such matrices define an associative commutative multiplication

$X_\mu \circ X_\mu \equiv \alpha^\sigma_{\mu\nu} X_\sigma$ on the subspace $\mathbf{V}$ and vice versa. For an example of this kind, let us take $(\alpha_\mu)^\sigma_\nu = \delta^\sigma_{\mu+\nu}$, where $\mu, \nu, \sigma = 0, \ldots, n$. Introduced thus, the α_μ form an Abelian matrix ring and define a semidirect sum $\mathbf{L} = \mathbf{H} \triangleleft \mathbf{V}$ with $\mathbf{H} \sim \mathbf{V}^*$. They comply with the condition of the theorem, since the matrix α_0, the unity of the ring, has a maximum rank equal to $n+1$, and we may set $\xi^\mu = \delta^0_\mu$ in the transformation (5).

3. The universal $\mathcal{R}$-matrix for the quantum Poincaré algebra

Let us apply the technique developed in section 2 to the quantum universal enveloping Poincaré algebra. The deformation is generated by the twisting of the subalgebra $\mathbf{L} = \mathrm{Span}(E_1, E_2, P_1, P_2, P_+, K_3)$, in terms of [1]. Introducing the notation $H^i = -zE_i$, $H^3 = -zP_+$, $Y_i = 2P_i$, $Y_3 = -2K_3$, $i = 1, 2$, the coproduct in $U_\alpha(\mathbf{L})$ reads

$$\Delta_\alpha(H^\mu) = H^\mu \otimes 1 + 1 \otimes H^\mu$$

$$\Delta_\alpha(Y_1) = \mathrm{e}^{H^3} \otimes Y_1 + Y_1 \otimes \mathrm{e}^{-H^3}$$

$$\Delta_\alpha(Y_2) = \mathrm{e}^{H^3} \otimes Y_2 + Y_2 \otimes \mathrm{e}^{-H^3}$$

$$\Delta_\alpha(Y_3) = \mathrm{e}^{H^3} \otimes Y_3 + Y_3 \otimes \mathrm{e}^{-H^3} + \mathrm{e}^{H^3} H^1 \otimes Y_1 - Y_1 \otimes H^1 \mathrm{e}^{-H^3} + \mathrm{e}^{H^3} H^2 \otimes Y_2 - Y_2 \otimes H^2 \mathrm{e}^{-H^3}.$$

The non-vanishing commutators are

$$[H^i, Y_i] = 2\sinh(H^3) \qquad [H^i, Y_3] = 2\cosh(H^3) H^i \qquad i = 1, 2$$

$$[H^3, Y_3] = 2\sinh(H^3).$$

Correspondence with the notation of the previous paragraph is achieved through the transformation $X_\mu = Y_\nu (\mathrm{e}^{\alpha \cdot H})^\nu_\mu$, the matrices α being

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \alpha_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Explicitly this results in the following change of variables:

$$X_1 = Y_1 \mathrm{e}^{H^3} \qquad X_2 = Y_2 \mathrm{e}^{H^3} \qquad X_3 = (Y_1 H^1 + Y_2 H^2 + Y_3) \mathrm{e}^{H^3}.$$

Transition to the classical generators is completed by the transformation (5), where we may assume $\xi^1 = \xi^2 = 0$, $\xi^3 = \frac{1}{2}$, as the matrix α_3 has a maximum rank of 3:

$$K^1 = H^1 \mathrm{e}^{-2H^3} \qquad K^2 = H^2 \mathrm{e}^{-2H^3} \qquad K^3 = \tfrac{1}{2}(1 - \mathrm{e}^{-2H^3}).$$

The elements K^μ and X_ν obey the ordinary, non-quantum, commutation relations of $U(\mathbf{L})$:

$$[K^i, X_i] = 2K^3 \qquad [K^i, X_3] = 2K^i \qquad i = 1, 2$$

$$[K^3, X_3] = 2K^3.$$

The quantization $U(\mathbf{L}) \to U_\alpha(\mathbf{L})$ is controlled by the twisting element

$$\mathcal{F} = \exp\left(\frac{K^1}{2K^3 - 1} \otimes X_1 + \frac{K^2}{2K^3 - 1} \otimes X_2 + \frac{1}{2} \ln(1 - 2K^3) \otimes X_3 \right) = \exp(-H^\mu \otimes X_\mu)$$

and the quantum universal $\mathcal{R}$-matrix of the algebra $U_\alpha(\mathbf{L})$ is given by

$$\mathcal{R} = \exp(X_\mu \otimes H^\mu) \exp(-H^\mu \otimes X_\mu).$$

In order to compare this result with the expression for the $\mathcal{R}$-matrix obtained in [3], let us write out Hopf operations in terms of generators $H^\mu,\ X_\mu$, explicitly:

$$\Delta_\alpha(H^\mu) = H^\mu \otimes 1 + 1 \otimes H^\mu \qquad \Delta_\alpha(X_i) = \mathrm{e}^{2H^3} \otimes X_i + X_i \otimes 1$$

$$\Delta_\alpha(X_3) = \mathrm{e}^{2H^3} \otimes X_3 + X_3 \otimes 1 + 2\mathrm{e}^{2H^3} H^1 \otimes X_1 + 2\mathrm{e}^{2H^3} H^2 \otimes X_2.$$

$$[H^i, X_i] = \mathrm{e}^{2H^3} - 1 \qquad [H^i, X_3] = \mathrm{e}^{2H^3} H^i \qquad i = 1, 2$$

$$[H^3, X_3] = \mathrm{e}^{2H^3} - 1.$$

Now note that, due to the comutation properties of the generators, the $\mathcal{R}$-matrix can be factorized:

$$\mathcal{R} = \mathrm{e}^{X_2 \otimes H^2} \mathrm{e}^{X_1 \otimes H^1} \mathrm{e}^{X_3 \otimes H^3} \mathrm{e}^{-H^3 \otimes X_3} \mathrm{e}^{-H^1 \otimes X_1} \mathrm{e}^{-H^2 \otimes X_2}.$$

Substitution of $H^3 = zP_+,\ H^i = zE_i,\ X_3 = -2K_3,\ X_i = 2P_i$ into this formula leads to the expression which differs from the $\mathcal{R}$-matrix of [3] by the permutation of the tensor components. This indicates that our results are in accordance with [3], because the above substitution provides the same multiplication but the opposite coproduct to those of [3].

4. Conclusion

The present investigation continues the series of works [7–9] devoted to a method of constructing quantum Lie algebras with the use of a classical object, the dual group. Based on the duality principle [12] viewing a quantum universal Lie algebra as a set of non-commutative functions on the dual group, that method reduces the quantizaton problem to finding a deformed Lie biideal consistent with a given coproduct [8]. Because of complicated structure of a generic Lie group, that programme is feasible, however, for simple classes or in separate particular cases. So is the set of quasi-Abelian groups which can be represented as a semidirect composition of two commutative subgroups. The quantization theory for such Lie algebras was developed in [8]. Simple as dual groups of that type may seem, they occur rather frequently, especially in low dimensions [1, 13–16], and the corresponding quantum algebras possess very diverse and non-trivial properties. So, that class includes two non-isomorphic deformations of $U(sl(2))$. A generalization of the standard quantization was studied in detail in our work [9]. Here we have focused on a generalization of the non-standard quantization of $sl(2)$ or, to be more exact, its Borel subalgebra. The universal $\mathcal{R}$-matrix for the Jordanian $U_h(sl(2))$ was found by Ogievetsky and Vladimirov [10, 11, 17] (see also [18]). In the present paper we have obtained explicit expression for $\mathcal{F}$ and $\mathcal{R}$ for the multi-dimensional generalization of the Borel subalgebra in $sl(2)$. The technique developed made it possible to built the twisting cocycle for the null-plane quantizaton of the Poincaré algebra $\mathcal{P}(3+1)$ and to give the interpretation for the universal $\mathcal{R}$-matrix found in [3] in the framework of the twisting theory.

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